# Two-dimensional lattice Boltzmann model for magnetohydrodynamics

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We present a lattice Boltzmann model for the simulation of two-dimensional magnetohydro dynamic (MHD) flows. The model is an extension of a hydrodynamic lattice Boltzman model with 9 velocities on a square lattice resulting in a model with 17 velocities. Earlier lattice Boltzmann models for two-dimensional MHD used a bidirectional streaming rule. However, the use of such a bidirectional streaming rule is not necessary. In our model, the standard streaming rule is used, allowing smaller viscosities. To control the viscosity and the resistivity independently, a matrix collision operator is used. The model is then applied to the Hartmann flow, giving reasonable results.

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## I. INTRODUCTION

During the last years, the lattice Boltzmann equation (LBE) method evolved to an alternative method for the simulation of fluid flows [1,2]. It originated from a Boolean fluid model, the so-called lattice-gas automata (LGA) [3,4]. LGA simulate fluids directly as a system of discrete particles moving on a regular lattice. The particles move from a lattice cell to one of its nearest neighbors. The particles undergo collisions at the lattice cells conserving mass and momentum. The macroscopic fluid variables such as density and flow speed are calculated by averaging over many lattice cells and time steps. A successful LGA model was the FHP gas, introduced by Frisch, Hasslacher, and Pomeau [5]. It uses a hexagonal lattice with six particle speeds. However, LGA have some shortcomings such as the lack of Galilean invariance and statistical noise. Instead of following the dynamics of individual particles, lattice Boltzmann methods use the single-particle distribution function which removes the statistical noise [6].

Shortly after the FHP gas, the first magnetohydrodynamic LGA was developed by Montgomery and Doolen [7,8]. Their model is an extension of the original FHP gas. It includes additional degrees of freedom of the particles for the vector potential which has only one component in two dimensions and satisfies a passive scalar equation similar to the temperature. Therefore, the model is confined to two dimensions. Additionally, the Lorentz force is not included automatically in the model. It must be included by hand as an external force which needs some space averages.

Another magnetohydrodynamic model was developed by Chen and Matthaeus and Chen, Matthaeus, and Klein [9,10]. In this model, a two-indexed particle distribution is used. Each state is associated with two velocity vectors of the FHP gas, i.e., there are 36 different states for the particles. During the streaming step, each particle moves along one of the two vectors which is chosen randomly. This model includes the Lorentz force by pure local operations and it is not confined to two dimensions.

This model has been extended to a lattice Boltzmann model by Chen *et al.* [11,12]. Although the model is not confined to two dimensions, its extension to three dimensions would require a large amount of computational memory. For $N$ moving directions, this model needs $N\times N$ particle states. For example, the face-centered-hypercubic (FCHC) model has 24 different particle speeds resulting in at least 576 particle states for a MHD model.

Martinez, Chen, and Matthaeus [13] reduced the number of necessary particle states from 37 to 13 in the two-dimensional case. This reduction makes a three-dimensional extension of the model possible. They applied the model to the Hartmann flow and two-dimensional magnetic reconnection in a sheet pinch. The comparison with a spectral method shows reasonable results. However, the main difficulty of this model is that it is confined to low-Reynolds numbers because the values of the transport coefficients at the stability threshold are finite. This is caused by the use of the bidirectional streaming rule. The transport coefficients of hydrodynamic lattice Boltzmann models vanish at the stability threshold, allowing, in principal, arbitrary high-Reynolds numbers.

Some other lattice Boltzmann models for MHD have been developed. Succi, Vergasola, and Benzi [14] developed a model which is a two-dimensional projection of the four-dimensional FCHC model. It is also confined to two dimensions. Fogaccia, Benzi, and Romanelli [15] presented a Lattice Boltzmann model for the simulation of three-dimensional plasma turbulence.

In this paper, we demonstrate that the use of the bidirectional streaming rule is not necessary for a MHD model. In Sec. II, the model is described in detail and the model equations are derived. In Sec. III, we present the test simulations of the Hartmann flow and in Sec. IV we give the conclusions. Finally, in Appendix A, we present some tensorial relations used for the derivation of the model equations and in Appendix B, we calculate the inverse of the collision matrix.

## II. MHD MODEL

### A. Lattice Boltzmann equation

The lattice Boltzmann equation describes the evolution of the particle distribution function $f_i$

$$f_i(\mathbf{x}+\mathbf{v}_i,t+1)-f_i(\mathbf{x},t)=\Omega_i, \tag{1}$$

where $\mathbf{x}$ are the lattice cells, $t$ is the discret time, $v_i$ are the velocities associated with the distribution function $f_i$, and $\Omega_i$ is the collision term. The density $\rho$ and the velocity $\mathbf{v}$ are obtained from the distribution function by

$$\rho=\sum_i f_i, \tag{2}$$

$$\rho\mathbf{v}=\sum_i f_i\mathbf{v}_i. \tag{3}$$

For LBE, the Bathnagar-Gross-Krook (BGK) [16] single time relaxation ansatz is often used for the collision term [17,18]

$$\Omega_i=-\tfrac{1}{\tau}(f_i-f_i^{\mathrm{eq}}), \tag{4}$$

where $f_i^{\mathrm{eq}}$ is the equilibrium distribution function. It depends on the lattice and it is usually not uniquely defined. For square and cubic lattices, the distribution function has the form [18]

$$f_i^{\mathrm{eq}}=w_i\rho[1+3\mathbf{v}_i\mathbf{v}+\tfrac{9}{2}(\mathbf{v}_i\mathbf{v})^2-\tfrac{3}{2}v^2]. \tag{5}$$

$w_i$ is a function of $|\mathbf{v}_i|$ and depends on the number of velocities included in the model. For two-dimensional simulations on a square lattice, a model with nine velocities, the so-called D2Q9 model is often used. The model includes four components $f_i$, $i=1,...,4$ with velocities $\mathbf{v}_i$ pointing to the nearest neighbors, four components $f_i$, $i=5,...,8$ with velocities $\mathbf{v}_i$ pointing to the next-nearest lattice cells and a component $f_0$ for rest particles with zero speed. The values of $w_i$ are 4/9, 1/9, and 1/36 for $|\mathbf{v}_i|$ equal to 0, 1, and $\sqrt{2}$.

### B. General description of a MHD model

In addition to their model on a hexagonal lattice, Martinez, Chen, and Matthaeus [13] presented a MHD model on a square lattice. Our model is very similar to their model except that our model does not use the bidirectional streaming. It is an extension of the D2Q9 model described in the previous section. Following Martinez, Chen and Matthaeus [13], we divide the nine velocities and corresponding components of the particle distribution function of the D2Q9 model into three groups. The first group contains the component $f_0$ with zero speed. The second group contains the four velocities pointing to the nearest neighbors which are now labeled as $\mathbf{v}_i^{\mathrm{I}}$, $i=1,...,4$ with $\mathbf{v}_i^{\mathrm{I}}=(\cos(i-1)\pi/2, \sin(i-1)\pi/2)$. The last group contains the velocities $\mathbf{v}_i^{\mathrm{II}}$, $i=1,...,4$ pointing to the next-nearest neighbors with $\mathbf{v}_i^{\mathrm{II}}=\sqrt{2}((\cos(i-1/2)\pi/2, \sin(i-1/2)\pi/2)$. The components of the particle distribution function in the second and third groups are divided into two subcomponents $f_{ij}^K$ where $K=\mathrm{I}$, II, $i=1,...,4$, and $j=i\pm1$ (mod 4). There are now 17 components of the particle distribution function: a distribution of rest particles $f_0$ and a streaming part $f_{ij}^K$. Each component of the streaming part $f_{ij}^K$ is associated with two vectors $\mathbf{v}_i^K$ and $\mathbf{v}_j^K$ where the component $f_{ij}^K$ of the particle distribution function propagates along $\mathbf{v}_i^K$. The density $\rho$, the velocity $\mathbf{v}$, and the magnetic-field $\mathbf{B}$ are defined as

$$\rho=f_0+\sum_{i,j,K} f_{ij}^K, \tag{6}$$

$$\rho\mathbf{v}=\sum_{i,j,K}\mathbf{v}_i^K f_{ij}^K, \tag{7}$$

$$\mathbf{B}=\sum_{i,j,K}\mathbf{v}_j^K f_{ij}^K. \tag{8}$$

Because we use the standard streaming rule with propagation along $v_i^K$, the streaming part of the particle distribution function $f_{ij}^K$ satisfies the lattice Boltzman equation

$$f_{ij}^K(\mathbf{x}+\mathbf{v}_i^K,t+1)-f_{ij}^K(\mathbf{x},t)=\Omega_{ij}^K. \tag{9}$$

For the nonstreaming part $f_0$ of the distribution function we have the equation

$$f_0(\mathbf{x},t+1)-f_0(\mathbf{x},t)=\Omega_0. \tag{10}$$

Instead of the BGK collision term, we use a matrix collision operator

$$\Omega_{ij}^K=-\tfrac{1}{\tau}(f_{ij}^K-f_{ij}^{K(\mathrm{eq})})+\theta M_{ijmn}^{KK'}(f_{mn}^{K'}-f_{mn}^{K'(\mathrm{eq})}) \tag{11}$$

$$\Omega_0=-\tfrac{1}{\tau}(f_0-f_0^{(\mathrm{eq})}). \tag{12}$$

The matrix $M_{ijmn}^{KK'}$ is given by

$$M_{ijmn}^{KK'}=T_{\alpha\beta\gamma\delta}v_{i\alpha}^K v_{j\beta}^K v_{m\gamma}^{K'} v_{n\delta}^{K'}. \tag{13}$$

The tensor $T_{\alpha\beta\gamma\delta}$ has the form

$$T_{\alpha\beta\gamma\delta}=-\tfrac{1}{32}\delta_{\alpha\beta}\delta_{\gamma\delta}+\tfrac{3}{20}\delta_{\alpha\gamma}\delta_{\beta\delta}+\tfrac{1}{10}\delta_{\alpha\delta}\delta_{\beta\gamma}-\tfrac{3}{16}\delta_{\alpha\beta\gamma\delta}, \tag{14}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta and $\delta_{\alpha\beta\gamma\delta}=1$ only if $\alpha=\beta=\gamma=\delta$ otherwise it is 0. Roman indices label the components of the particle distribution function and Greek indices label the spatial dimensions. For the latter, the Einstein summation convention is used. The matrix collision operator allows an independent control over the viscosity and the resistivity.

The next step is the specification of the equilibrium distribution function $f_{ij}^{\mathrm{eq}}$ in such a way that the model reproduces the correct MHD equations. In addition it must be compatible with the definitions (6)–(8). A possible form of the equilibrium distribution function is

$$f_0^{(\mathrm{eq})}=w_0[\rho(1-\tfrac{3}{2}v^2)-\tfrac{3}{8}B^2], \tag{15}$$

$$f_{ij}^{K(\mathrm{eq})}=w_K\{\rho[1+3\mathbf{v}_i^K\mathbf{v}+\tfrac{9}{2}(\mathbf{v}_i^K\mathbf{v})^2-\tfrac{3}{2}v^2]+3\mathbf{v}_j^K\mathbf{B}-\tfrac{9}{2}(\mathbf{v}_i^K\mathbf{B})^2 +3B^2+\tfrac{9}{4}(\mathbf{v}_j^K\times\mathbf{v}_i^K)(\mathbf{B}\times\mathbf{v})\}, \tag{16}$$

where the first term is the hydrodynamic part which has the same form as for the D2Q9 model and the weighting factors are $w_0=4/9$, $w_{\mathrm{I}}=1/18$, and $w_{\mathrm{II}}=1/72$.

### C. Model equations

Now, the model equations can be derived from the lattice Boltzmann Eq. (9) using a Chapman-Enskog expansion [4]. First, we expand the left-hand side of Eq. (9) up to second order in space and time

$$\partial_t f_{ij}^K(\mathbf{x},t)+v_{i\beta}^K\partial_\beta f_{ij}^K+\tfrac{1}{2}\partial_t^2 f_{ij}^K+v_{i\beta}^K\partial_\beta\frac{\partial f_{ij}^K}{\partial t}+\tfrac{1}{2}v_{i\beta}^K v_{i\gamma}^K\partial_\beta\partial_\gamma f_{ij}^K=\Omega_{ij}^K.$$

Next, we expand the particle distribution in a series of powers of a small parameter $\epsilon$

$$f_{ij}^K=f_{ij}^{K(0)}+\epsilon f_{ij}^{K(1)}+\cdots, \tag{17}$$

where $f_{ij}^{K(0)}=f_{ij}^{K(\mathrm{eq})}$ and the higher terms in $\epsilon$ are the departure from the local equilibrium. These higher terms of the particle distribution function do not contribute to the macroscopic variables such as density and momentum, i.e.,

$$f_0^{(1)}+\sum_{i,j,K}f_{ij}^{K(1)}=0, \tag{18}$$

$$\sum_{i,j,K}\mathbf{v}_i^K f_{ij}^{K(1)}=0, \tag{19}$$

$$\sum_{i,j,K}\mathbf{v}_j^K f_{ij}^{K(1)}=0. \tag{20}$$

The time and space derivations are also expanded in powers of $\epsilon$,

$$\partial_t=\epsilon\partial_{t1}+\epsilon^2\partial_{t2}, \tag{21}$$

$$\partial_\beta=\epsilon\partial_{\beta 1}. \tag{22}$$

$t1$ captures the fast changes, for example, sound waves, whereas $t2$ is associated with the slower dissipative processes. Inserting the expansions (17), (21), and (22) into Eq. (17) we get up to first order in $\epsilon$

$$\partial_{t1}f_{ij}^{K(0)}+v_{i\beta}^K\partial_\beta f_{ij}^{K(0)}=A_{ijmn}^{KK'}f_{mn}^{K'(1)}, \tag{23}$$

where we introduced the collision matrix

$$A_{ijmn}^{KK'}=-\tfrac{1}{\tau}\delta_{im}\delta_{jn}\delta_{KK'}+\theta M_{ijmn}^{KK'}. \tag{24}$$

Multiplying Eq. (23) with 1, $v_{i\alpha}$, and $v_{j\alpha}$ and summing over $i$, $j$, and $K$ one gets

$$\partial_{t1}\rho+\partial_{\beta 1}(\rho v_\beta)=0, \tag{25}$$

$$\partial_{t1}(\rho v_{i\alpha})+\partial_{\beta 1}\Pi_{\alpha\beta}^{(0)}=0, \tag{26}$$

$$\partial_{t1}B_\alpha+\partial_{\beta 1}\Lambda_{\alpha\beta}^{(0)}=0, \tag{27}$$

where $\Pi_{\alpha\beta}^{(0)}=\Sigma_{i,j,K}f_{ij}^{K(0)}v_{i\alpha}^K v_{i\beta}^K$ is the momentum flux tensor and $\Lambda_{\alpha\beta}^{(0)}=\Sigma_{i,j,K}f_{ij}^{K(0)}v_{j\alpha}^K v_{i\beta}^K$ is the magnetic momentum flux tensor. The right-hand sides vanish because the collision term conserves the macroscopic variables. Unsing Eq. (16) for the equilibrium distribution function and the tensor relations of Appendix A, we get

$$\Pi_{\alpha\beta}^{(0)}=\tfrac{1}{3}\rho\delta_{\alpha\beta}+\rho v_\alpha v_\beta+\tfrac{1}{2}B^2\delta_{\alpha\beta}-B_\alpha B_\beta, \tag{28}$$

$$\Lambda_{\alpha\beta}^{(0)}=B_\alpha v_\beta-B_\beta v_\alpha. \tag{29}$$

Up to first order in $\epsilon$ we get the following equations:

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0, \tag{30}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right]=-\boldsymbol{\nabla}\left(p+\frac{B^2}{2}\right)+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}+\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{B}), \tag{31}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}), \tag{32}$$

with the equation of state $p=\rho/3$. To get the dissipative effects, we have to consider the second-order terms

$$\partial_{t2}f_{ij}^{K(0)}+\tfrac{1}{2}\partial_{t1}^2 f_{ij}^{K(0)}+\partial_{t1}v_{i\beta}^K\partial_{\beta 1}f_{ij}^{K(0)}\tfrac{1}{2}v_{i\beta}^K v_{i\gamma}^K\partial_{\beta 1}\partial_{\gamma 1}f_{ij}^{K(0)}+\partial_{t1}f_{ij}^{K(1)}+v_{i\beta}^K\partial_{\beta 1}f_{ij}^{K(1)}=A_{ijmn}^{KK'}f_{mn}^{K'(2)}. \tag{33}$$

First, we calculate the change of the density $\partial_{t2}\rho$ due to dissipative processes. Summing over the velocity states in Eq. (33) we get

$$\partial_{t2}\rho+\tfrac{1}{2}\partial_{t1}[\partial_{t1}\rho+\partial_{\beta 1}(\rho v_\beta)]+\tfrac{1}{2}\partial_{\beta 1}(\partial_{t1}(\rho v_\beta)+\partial_{\gamma 1}\Pi_{\beta\gamma}^{(0)})+\partial_{t1}\sum_{i,j,K}f_{ij}^{K(1)}+\partial_{\beta 1}\sum_{i,j,K}v_\beta f_{ij}^{K(1)}=0. \tag{34}$$

Using Eqs. (25)–(27) and Eqs. (18–20) we get

$$\partial_{t2}\rho=0. \tag{35}$$

There are no second-order terms in the continuity equation. Next, we calculate $\partial_{t2}(\rho v_\alpha)$. Multiplying Eq. (33) by $v_{i\alpha}$ and summing over the velocity states, we get

$$\partial_{t2}(\rho v_\alpha)+\tfrac{1}{2}\partial_{t1}(\partial_{t1}(\rho v_\alpha)+\partial_{\beta 1}\Pi_{\alpha\beta}^{(0)})+\tfrac{1}{2}\partial_{\beta 1}\left(\partial_{t1}\Pi_{\alpha\beta}^{(0)}+\partial_{\gamma 1}\sum_{i,j,K}v_{i\alpha}^K v_{i\beta}^K v_{i\gamma}^K f_{ij}^{K(0)}\right)+\partial_{t1}\sum_{i,j,K}v_{i\alpha}f_{ij}^{K(1)}+\partial_{\beta 1}\sum_{i,j,K}v_{i\alpha}^K v_{i\beta}^K f_{ij}^{K(1)}=0. \tag{36}$$

Again using Eqs. (25)–(27) and Eqs. (18–20) this becomes

$$\partial_{t2}(\rho v_\alpha)=-\tfrac{1}{2}\partial_{\beta 1}\left(\partial_{t1}\Pi_{\alpha\beta}^{(0)}+\partial_{\gamma 1}\sum_{i,j,K}v_{i\alpha}^K v_{i\beta}^K v_{i\gamma}^K f_{ij}^{K(0)}\right)-\partial_{\beta 1}\sum_{i,j,K}v_{i\alpha}^K v_{i\beta}^K f_{ij}^{K(1)}. \tag{37}$$

We calculate $f_{ij}^{K(1)}$ from Eq. (23) using the inverse collision matrix of Appendix B and get

$$f_{ij}^{K(1)} = -\tau(\partial_{t1} f_{ij}^{K(0)} + v_{i\beta}\partial_{\beta 1} f_{ij}^{K(0)}) + \frac{\theta\tau^2}{\theta\tau - 1} M_{ijmn}^{KK'}(\partial_{t1} f_{mn}^{K'(0)} + v_{m\beta}\partial_{\beta 1} f_{mn}^{K'(0)}). \quad (38)$$

Inserting this into Eq. (37) we get

$$\partial_{t2}(\rho v_\alpha) = \partial_{\beta 1}\left\{ (\tau - \tfrac{1}{2}) \left( \partial_{t1}\Pi_{\alpha\beta}^{(0)} + \partial_{\gamma 1} \sum_{i,j,K} v_{i\alpha}^K v_{i\beta}^K v_{i\gamma}^K f_{ij}^{K(0)'} \right) + \frac{\theta\tau^2}{1-\theta\tau} v_{i\alpha}^K v_{i\beta}^K M_{ijmn}^{KK'}(\partial_{t1} f_{mn}^{K'(0)} + \partial_{\gamma 1} v_{m\gamma}^{K'} f_{mn}^{K'(0)}) \right\}. \quad (39)$$

Using the tensor relations of Appendix A, we see that there is no contribution to the dissipative term of the momentum equation due to the matrix $M_{ijmn}^{K\text{-}K'}$ and the other terms reduce to

$$\partial_{t2}(\rho v_\alpha) = \tfrac{1}{3}(\tau - \tfrac{1}{2})\partial_{\beta 1}[\partial_{\alpha 1}(\rho v_\beta) + \partial_{\beta 1}(\rho v_\alpha)]. \quad (40)$$

A similar calculation gives the dissipative term in the induction equation

$$\partial_{t2} B_\alpha = \left(\tau - \frac{1}{2} + \frac{\theta\tau^2}{1-\theta\tau}\right)\partial_{\beta 1}\left(\partial_{\beta 1} B_\alpha - \frac{2}{3}\partial_{\alpha 1} B_\beta\right). \quad (41)$$

Using Eqs. (21) and (22) we combine the first- and second-order terms to the following equations in vector form:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \quad (42)$$

$$\rho\left[\frac{\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla\left(p + \frac{B^2}{2}\right) + (\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{B}) + \tfrac{1}{3}[(\tau - \tfrac{1}{2})(\Delta\rho\mathbf{v} + \nabla(\nabla\cdot\rho\mathbf{v}))], \quad (43)$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \left(\tau - \frac{1}{2} + \frac{\theta\tau^2}{1-\theta\tau}\right)(\mathbf{\Delta B} - \tfrac{2}{3}\nabla(\nabla\cdot\mathbf{B})). \quad (44)$$

This set of equations does not include the divergence-free condition of the magnetic field. Building the divergence of the induction equation, we see that the divergence of the magnetic field satisfies a diffusion equation

$$\frac{\partial\nabla\cdot\mathbf{B}}{\partial t} = \frac{1}{3}\left(\tau - \frac{1}{2} + \frac{\theta\tau^2}{1-\theta\tau}\right)\Delta\nabla\cdot\mathbf{B}. \quad (45)$$

The solenoidal part of the magnetic field diffuses away. If the magnetic field is divergence free at the beginning, it is divergence free for all time, i.e., the divergence-free condition can be added to the model as an initial condition. If the flow speed is small compared to the speed of sound $c_s = 1/\sqrt{3}$, the fluid can be treated as incompressible. Additionally, the magnetic field must be small enough that the magnetic pressure is negligible. For an incompressible fluid and a divergence-free magnetic field,

$$\nabla\cdot\mathbf{v} = 0, \quad (46)$$

$$\nabla\cdot\mathbf{B} = 0, \quad (47)$$

the model equations simplify to

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p + \frac{B^2}{2}\right) + \tfrac{1}{\rho}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\Delta\mathbf{v}, \quad (48)$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \mu\mathbf{\Delta B}, \quad (49)$$

where $\nu$ is the kinematic viscosity and $\mu$ is the resistivity. The values of the transport coefficients are

$$\nu = \tfrac{1}{3}(\tau - \tfrac{1}{2}), \quad (50)$$

$$\mu = \tau - \frac{1}{2} + \frac{\theta\tau^2}{1-\theta\tau}. \quad (51)$$

They can be controlled independently by the parameters $\tau$ and $\theta$. There is no lower bound of the transport coefficients because they vanish at the stability threshold $\tau = 1/2$ as in the hydrodynamic model for $\theta = 0$.

## III. HARTMANN FLOW

Next, we apply our model to a very simple MHD problem, the Hartmann flow [19]. This is the stationary flow of an incompressible, conductive fluid between two plates. Between the two plates, there is a homogenous magnetic field perpendicular to them. The direction of the initial magnetic field is chosen as the $y$ axis. We assume that the velocity of the fluid has only one component along the $x$-axis $\mathbf{v} = (v_x,0,0)$. The two plates are located at $y = L$ and $y = -L$. The flow produces an additional magnetic field. This field also has only an $x$ component. The total magnetic field is $\mathbf{B} = (B_x, B_0, 0)$, where $B_0$ is the constant strength of the initial magnetic field. From the continuity equation and the divergence-free condition of the magnetic field we get $\partial v_x/\partial x = 0$ and $\partial B_x/\partial x = 0$. Therefore, the nonlinear terms in the Navier-Stokes equation reduce to $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$ and $(\mathbf{B}\cdot\nabla)\mathbf{B} = (B_0\partial B_x/\partial y, 0, 0)$. The $y$ component and the $z$ component of the Navier-Stokes equation reduce to

$$\frac{\partial}{\partial y}\left(p + \frac{B^2}{2}\right) = 0, \quad (52)$$

$$\frac{\partial}{\partial z}\left(p + \frac{B^2}{2}\right) = 0. \quad (53)$$

$p + B^2/2$ depends only on $x$. The $x$ component of the Navier-Stokes equation is

$$\frac{1}{\rho}\frac{\partial}{\partial x}\left(p+\frac{B^2}{2}\right)=\frac{B_0}{\rho}\frac{\partial B_x}{\partial y}+\nu\left(\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right). \qquad (54)$$

Because the left-hand side depends only on $x$ and the right-hand side depends only on $y$ and $z$, both sides of the equation must be constant. We set this constant equal to $(1/\rho)\partial/\partial x(p+B^2/2)=-g$. This pressure gradient drives the flow. The nonlinear term in the induction equation reduces to $\nabla\times(\mathbf{v}\times\mathbf{B})=(B_0\partial v_x/\partial y,0,0)$. The $x$ component of the induction equation is

$$B_0\frac{\partial v_x}{\partial y}+\mu\left(\frac{\partial^2 B_x}{\partial y^2}+\frac{\partial^2 B_x}{\partial z^2}\right)=0. \qquad (55)$$

Assuming that all variables depend only on $y$, these equations reduce to the linear ordinary differential equations

$$\frac{B_0}{\rho}\frac{dB_x}{dy}+\nu\frac{d^2 v_y}{dy^2}=-g, \qquad (56)$$

$$B_0\frac{dv_x}{dy}+\mu\frac{d^2 B_x}{dy^2}=0. \qquad (57)$$

The velocity and the $x$ component of the magnetic field vanish at the plates

$$v_x(y)=0 \quad \text{at} \quad y=\pm L, \qquad (58)$$

$$B_x(y)=0 \quad \text{at} \quad y=\pm L. \qquad (59)$$

The solution of Eqs. (56) and (57) with the Boundary conditions (58) and (59) is

$$B_x(y)=\frac{\rho g L}{B_0}\left[\frac{\sinh(Hy/L)}{\sinh H}-\frac{y}{L}\right], \qquad (60)$$

$$v_x(y)=\sqrt{\frac{\rho\mu}{\nu}}\frac{gL}{B_0}\cosh H\left[1-\frac{\cosh(Hy/L}{\cosh H}\right], \qquad (61)$$

where $H=B_0L/\sqrt{\mu\nu\rho}$ is the Hartmann number which is the ratio between the magnetic and the viscous forces. In the limit of zero Hartmann number (no external field) the solution reduces to $B_x=0$ and $v_x=(gL^2/2\nu)(1-y^2/L^2)$ which is the parabolic velocity profile of the Poiseuille flow. For large Hartmann numbers, the velocity profile is flattened. The velocity is almost constant between the two plates except in a thin boundary layer of thickness $\delta=L/H$ where the velocity rises from zero to the constant value $v_0=\sqrt{\rho\mu/\nu}(gL/B_0)$.

Now, we present the results of the test simulation. The initial condition of the system was an equilibrium state with constant density $\rho=1$ and an uniform magnetic-field $B_0$ in the $y$ direction. The velocity and the $x$ component of the magnetic field was zero at the start of the simulation. The boundary conditions were realized by the method of Inamuro, Yoshino, and Ogino [20], adopted for MHD. For the unknown components of the particle distribution function at the boundary, an equilibrium distribution function with the parameters $\rho'$, $\mathbf{v}'$, and $\mathbf{B}'$ is chosen. These parameters are set such that the boundary conditions are fulfilled. The evolution of the system was computed until a steady state was reached. To drive the flow, we used an external force instead of a pressure gradient. The external force was realized by replacing $\mathbf{v}$ in the equilibrium distribution function at the right-hand side of the lattice Boltzmann equation by $\mathbf{v}+\tau\mathbf{g}$, where $\mathbf{g}$ is the acceleration of the fluid due to the external force. For all simulations, we used an array of $50\times3$ cells. The other parameters are set to $\tau=0.6$, $\theta=0$, and $g=10^{-6}$. The Hartmann number was varied by changing the strength of the constant vertical magnetic-field $B_0$. We simulated the Hartmann flow for $H=0$, 1, 2, 5, 10, and 20. Figure 1 shows the resulting velocity profiles and in Fig. 2, the correspond-

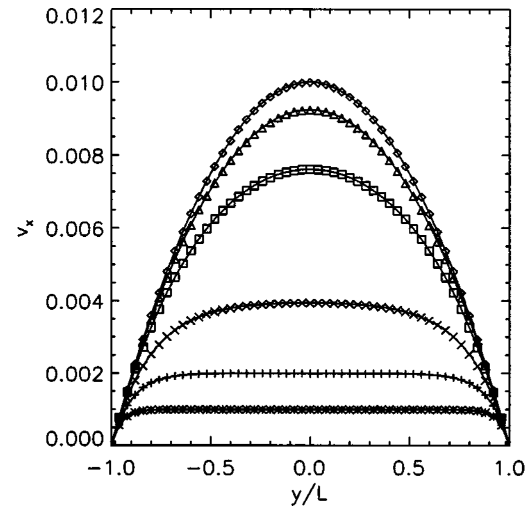


FIG. 1. Velocity profile $v_x$ vs $y/L$ for different Hartmann numbers: $H=0$ (diamonds), $H=1$ (triangles), $H=2$ (squares), $H=5$ (crosses), $H=10$ (plus signs), and $H=20$ (asterisks). The solid lines are the theoretical results.

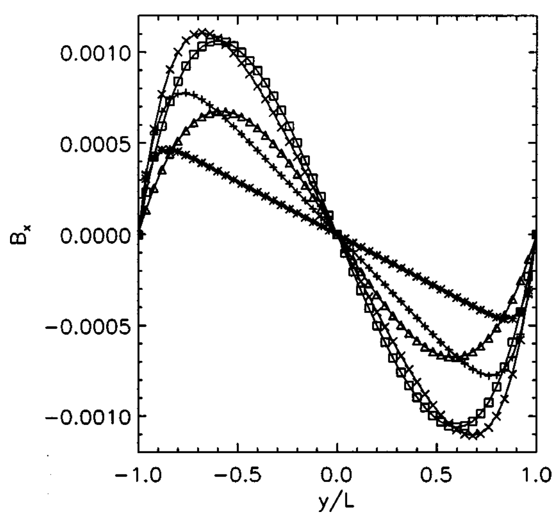


FIG. 2. Horizontal magnetic-field $B_x$ vs $y/L$ for different Hartmann numbers: $H=1$ (triangles), $H=2$ (squares), $H=5$ (crosses), $H=10$ (plus signs), and $H=20$ (asterisks). The solid lines are the theoretical results.

ing profiles of the horizontal magnetic-field $B_X$ are shown. The lines are the analytic solutions (60) and (61). The simulation results are in good agreement with the analytical solutions.

## IV. CONCLUSIONS

In this paper, we presented a LBE model for two-dimensional, incompressible MHD flows on a square lattice. In contrast to earlier MHD models, it uses the standard streaming rule. There is no need for using the bidirectional streaming rule of the earlier LGA and LBE MHD models. The flexibility of the LBE method allows us to get the correct form of the induction equation by an appropriate choice of the equilibrium distribution function only. The use of the standard streaming rule removes the lower bounds of the transport coefficients which appeared in the model of Martinez, Chen, and Matthaeus. Additionally, we extended the single time relaxation BGK collision term to a matrix collision term allowing independent control of the transport coefficients.

We applied the model to the Hartmann flow. The model gives accurate results for different values of the Hartmann number and the transport coefficients. This demonstrates that the model shows the correct MHD behavior. The model should also be able to simulate highly turbulent MHD flows. As the other MHD models based on the lattice Boltzmann equation, the divergenceless property of the magnetic field is not included in our model. However, this is not a real problem, because this property can be added as an initial condition for the magnetic field. Although, the model reproduces the Hartmann flow, future applications of the model on other more complicated MHD flows are needed to show its usability. The extension of the model to three dimension is in progress.

## APPENDIX A: SOME TENSOR RELATIONS

Here, we present the tensor relations [13] that have been used for the derivation of the model equations in Sec. II

$$\sum_{i,j} v^K_{i\alpha} = \sum_{i,j} v^K_{j\alpha} = 0, \tag{A1}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} = \sum_{i,j} v^K_{j\alpha} v^K_{j\beta} = 2A_K \delta_{\alpha\beta}, \tag{A2}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{j\beta} = 0, \tag{A3}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} v^K_{i\gamma} = \sum_{i,j} v^K_{j\alpha} v^K_{j\beta} v^K_{j\gamma} = 0, \tag{A4}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} v^K_{j\gamma} = \sum_{i,j} v^K_{i\alpha} v^K_{j\beta} v^K_{j\gamma} = 0, \tag{A5}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} v^K_{i\gamma} v^K_{i\delta} = \sum_{i,j} v^K_{j\alpha} v^K_{j\beta} v^K_{j\gamma} v^K_{j\delta} = 2Z_K \Delta_{\alpha\beta\gamma\delta} + 2Y_K \delta_{\alpha\beta\gamma\delta}, \tag{A6}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} v^K_{i\gamma} v^K_{j\delta} = \sum_{i,j} v^K_{i\alpha} v^K_{j\beta} v^K_{j\gamma} v^K_{j\delta} = 0, \tag{A7}$$

$$\sum_{i,j} v^K_{i\alpha} v^K_{i\beta} v^K_{j\gamma} v^K_{j\delta} = A_K^2 \delta_{\alpha\beta}\delta_{\gamma\delta} - 2Z_K \Delta_{\alpha\beta\gamma\delta} - 2Y_K \delta_{\alpha\beta\gamma\delta}, \tag{A8}$$

where $\Delta_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}$, $\delta_{\alpha\beta\gamma\delta} = 1$ only if $\alpha = \beta = \gamma = \delta$, otherwise it is 0, and $A_{\mathrm{I}} = 2$, $A_{\mathrm{II}} = 4$, $Z_{\mathrm{I}} = 0$, $Z_{\mathrm{II}} = 4$, $Y_{\mathrm{I}} = 2$, $Y_{\mathrm{II}} = -8$. With the above relations, one can calculate the lattice tensors including the weighting factors

$$\sum_{i,j,K} w_K v^K_{i\alpha} v^K_{i\beta} = \tfrac{1}{3} \delta_{\alpha\beta}, \tag{A9}$$

$$\sum_{i,j,K} w_K v^K_{i\alpha} v^K_{i\beta} v^K_{i\gamma} v^K_{i\delta} = \tfrac{1}{9} \delta_{\alpha\beta\gamma\delta}, \tag{A10}$$

$$\sum_{i,j,K} w_K v^K_{i\alpha} v^K_{i\beta} v^K_{j\gamma} v^K_{j\delta} = \tfrac{4}{9} \delta_{\alpha\beta}\delta_{\gamma\delta} - \tfrac{1}{9} \Delta_{\alpha\beta\gamma\delta}. \tag{A11}$$

## APPENDIX B: INVERSION OF MATRIX COLLISION OPERATOR

Our MHD model uses the following collision term:

$$\Omega^K_{ij} = A^{KK'}_{ijmn} (f^{K'}_{mn} - f^{K'(\mathrm{eq})}_{mn}) \tag{B1}$$

with the collision matrix

$$A^{KK'}_{ijmn} = -\tfrac{1}{\tau} \delta_{im}\delta_{jn}\delta_{KK'} + \theta M^{KK'}_{ijmn}. \tag{B2}$$

The matrix $M^{KK'}_{ijmn}$ is given by

$$M^{KK'}_{ijmn} = T_{\alpha\beta\gamma\delta} v^K_{i\alpha} v^K_{j\beta} v^{K'}_{m\gamma} v^{K'}_{n\delta}. \tag{B3}$$

The tensor $T_{\alpha\beta\gamma\delta}$ is given by

$$T_{\alpha\beta\gamma\delta} = -\tfrac{1}{32} \delta_{\alpha\beta}\delta_{\gamma\delta} + \tfrac{3}{20} \delta_{\alpha\gamma}\delta_{\beta\delta} + \tfrac{1}{10} \delta_{\alpha\delta}\delta_{\beta\gamma} - \tfrac{3}{16} \delta_{\alpha\beta\gamma\delta}. \tag{B4}$$

Multiplying the collision term with 1, $v^K_{i\epsilon}$, $v^K_{j\epsilon}$, summing over $i,j,K$ and using the tensor relations of Appendix A, we get

$$\sum_{i,j,K} \Omega^K_{ij} = 0, \tag{B5}$$

$$\sum_{i,j,K} \Omega^K_{ij} \mathbf{v}^K_i = \mathbf{0}, \tag{B6}$$

$$\sum_{i,j,K} \Omega^K_{ij} \mathbf{v}^K_j = \mathbf{0}. \tag{B7}$$

The collision operator conserves mass, momentum, and the magnetic field. The calculation of the transport coefficients requires the inversion of the collision matrix. This can easily be done if the matrix $M_{ijmn}^{KK'}$ satisfies

$$M_{ijmn}^{KK'}=\sum_{k,l,K''} M_{ijkl}^{KK''} M_{klmn}^{K''K'}. \tag{B8}$$

This holds if the Tensor $T_{\alpha\beta\gamma\delta}$ has the property

$$T_{\alpha\beta\gamma\delta}=\Theta_{\alpha\beta\epsilon\zeta} T_{\epsilon\zeta\eta\theta} \Theta_{\eta\theta\gamma\delta}, \tag{B9}$$

where the tensor $\Theta_{\alpha\beta\gamma\delta}$ is defined as

$$\Theta_{\alpha\beta\gamma\delta}=\sum_{i,j,K} v_{i\alpha}^K v_{j\beta}^K v_{i\gamma}^K v_{j\delta}^K$$
$$=-8\,\delta_{\alpha\beta}\delta_{\gamma\delta}+12\delta_{\alpha\gamma}\delta_{\beta\delta}-8\,\delta_{\alpha\delta}\delta_{\beta\gamma}+12\delta_{\alpha\beta\gamma\delta}. \tag{B10}$$

This can be verified directly. Using Eq. (B8), the inverse of the collision matrix can be calculated to

$$(A_{ijmn}^{KK'})^{-1}=-\tau\delta_{im}\delta_{jn}\delta_{KK'}+\frac{\theta\tau^2}{\theta\tau-1} M_{ijmn}^{KK'}. \tag{B11}$$


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